

# Five-dimensional Monopole Equation with Hedge-Hog Ansatz and Abel's Differential Equation

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## Abstract

We review the generalized monopole in the five-dimensional Euclidean space. A numerical solution with the Hedge-Hog ansatz is studied. The Bogomol'nyi equation becomes a second order autonomous non-linear differential equation. The equation can be translated into the Abel's differential equation of the second kind and is an algebraic differential equation.

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## I. TCHRAKIAN MONOPOLE IN FIVE DIMENSIONAL SPACE

The Dirac's magnetic monopole has charmed many people for sixty years [1]. There are a lot of papers on the magnetic monopole though nobody find it up to now. The Dirac's monopole is described as a configuration of gauge potential which is singular at a point. Because of the singularity, its mass or energy cannot be determined. t' Hooft and Polyakov realized the monopole as a classical solution of a non-Abelian gauge theory with adjoint scalar fields [2, 3]. Their solution has finite energy.

Yang considered the generalization of the Dirac monopole in five-dimensional space [4] and Tchrakian [5] found various generalizations of the t' Hooft-Polyakov monopole and the Belavin-Polyakov-Tyupkin-Schwartz instanton [6].

In this article, we focus on the five-dimensional monopole in one of the Tchrakian's models. The monopole is a classical configuration of the  $SO(5)$  five-dimensional gauge theory with scalar fields which form a vector representation of  $SO(5)$  with quartic terms with respect to the field strength. We need quartic terms to make the Bogomol'nyi completion [7]. In order to stabilize the asymptotic behavior of scalar fields, we only consider models which have the Higgs potential. We just consider equations after taking the Prasad-Sommerfeld limit [8]. We studied a Hedge-Hog solution of this model numerically [9]. The Bogomol'nyi equations of this model are non-linear ordinary equations. These equations are reduced to one second-order equation. The equation is autonomous and we obtain a first-order equation. This equation is the Abel's ordinary differential equation of the second kind [10, 11].

Let us consider the  $SO(5)$  gauge theory with fundamental matter in five-dimensional Euclidean space, whose action or pseudo energy is

$$E = \int \text{Tr} \left[ \frac{1}{8 \cdot 4!} F^{\wedge 2} \wedge *(F^{\wedge 2}) + \frac{1}{8} D_A \phi \wedge *(D_A \phi) + \lambda V(\phi) d^5 x \right].$$

Here  $V(\phi)$  is the Higgs potential of scalar field and we take the Prasad-Sommerfeld limit  $\lambda \rightarrow 0$ . The symbol “ $*$ ” represents the Hodge dual operator on the five-dimensional space with respect to the Euclidean metric. In this notation, we represent all multiplets in terms of the Clifford algebra which associates with the five-dimensional Euclidean metric. The Clifford algebra is generated by the Dirac's gamma matrices  $\gamma_a$ , ( $a = 1, 2, 3, 4, 5$ ). They satisfy the anti-commutation relation:  $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$ . These matrices are constructed by tensor products of the Pauli matrices  $\sigma_a$  whose multiplications is summarized in one equation:  $\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i\epsilon_{ijk} \sigma_k$  ( $i, j, k = 1, 2, 3$ ). Matrices  $\gamma_1 = \sigma_1 \otimes \mathbf{1}$ ,  $\gamma_2 = \sigma_2 \otimes \mathbf{1}$ ,  $\gamma_3 = \sigma_3 \otimes \sigma_1$ ,  $\gamma_4 = \sigma_3 \otimes \sigma_2$ ,  $\gamma_5 = -\sigma_3 \otimes \sigma_3$  satisfy the anti-commutation relation and an additional relation  $\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 = 1$ . Each  $\gamma_a$  is a Hermitian matrix. The generators of  $SO(5)$  are represented as commutators of  $\gamma_a$ ,  $\gamma_{ab} = (1/2)[\gamma_a, \gamma_b]$ . Let  $k$  be an integer and  $\mathfrak{S}_k$  be a symmetric group which consists of permutations of  $k$  characters. Let us define the anti-symmetric product of gamma matrices:  $\gamma_{m(1)\dots m(k)} := (1/k!) \sum_{\tau \in \mathfrak{S}_k} \gamma_{m(\tau(1))} \cdots \gamma_{m(\tau(k))}$ . We represent the gauge potential as  $A = (1/2)A_m^{ab} \gamma_{ab} dx^m$  and the scalar field is also represented as  $4 \times 4$  matrices  $\phi = \phi^a \gamma_a$ . The field strength 2-form is given by  $F = dA + gA^2$  where  $g$  is the gauge coupling constant. A covariant derivative 1-form of  $\phi$  is given by  $D_A \phi = d\phi + g[A, \phi]$ . The  $N$ -th power of a differential form  $\omega$  is represented as  $\omega^{\wedge N} := \omega \wedge \omega \wedge \cdots \wedge \omega$ . Let us use the notation  $dx^{m_1 m_2 \dots m_n} = dx^{m_1} \wedge dx^{m_2} \wedge \cdots \wedge dx^{m_n}$ . All two elements in  $\{dx^{mn}\}$  commute with each other:  $dx^{mn} \wedge dx^{pq} = dx^{pq} \wedge dx^{mn}$ . In order to explain the square of the field strength in terms of its components, we show the anti-commutation relation of  $\gamma_{ab}$ :  $\{\gamma_{ab}, \gamma_{cd}\} = 2\gamma_{abcd} + 2(\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd})$ . The relation  $\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 = 1$  implies that  $\gamma_{abcd} = \epsilon_{abcde} \gamma_e$ , where  $\epsilon_{abcde}$  is the five-dimensional Levi-Civita anti-symmetric tensor such that  $\epsilon_{12345} = 1$ . The Kronecker delta on the rank-two anti-symmetric tensor is  $\delta_{cd}^{[ab]} = (1/2)(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$ . This tensor satisfies the relation  $\delta_{cd}^{[ab]} T^{cd} = T^{ab}$  for all anti-symmetric tensor  $T^{cd}$ . Let us rewrite the pseudo-energy in terms of components. The square of the field strength  $F = (1/4)F_{mn}^{ab} \gamma_{ab} dx^{mn}$  is  $F \wedge F = (1/16)F_{mn}^{ab} F_{pq}^{cd} (\epsilon_{abcde} \gamma_e - 2\delta_{cd}^{[ab]}) dx^{mnpq}$ . The components should be anti-symmetrized with respect to the spatial indices because the differential form  $dx^{mnpq}$  is anti-symmetric. We define two four-rank anti-symmetric tensors  $T_{mnpq}^e = (1/2 \cdot 4!) \epsilon_{abcde} (F_{mn}^{ab} F_{pq}^{cd} + F_{mp}^{ab} F_{qn}^{cd} + F_{mq}^{ab} F_{np}^{cd})$  and  $S_{mnpq} = (1/4!) (F_{mn}^{ab} F_{pq}^{ab} + F_{mp}^{ab} F_{qn}^{ab} + F_{mq}^{ab} F_{np}^{ab})$ . With these tensors,  $F^{\wedge 2} = (T_{mnpq}^e \gamma_e - S_{mnpq}) dx^{mnpq}$ . The gauge part of the density of the pseudo-energy is bilinear with respect to these tensors  $T_{mnpq}^e, S_{mnpq}$ :  $(1/4) \text{Tr} F^{\wedge 2} \wedge *(F^{\wedge 2}) = (T_{mnpq}^e T_{mnpq}^e + S_{mnpq} S_{mnpq}) dv$  where  $dv$  is the volume form  $dx^{12345}$ . The covariant derivative of the scalar field is  $D_A \phi = (D_A \phi)_m^a \gamma_a dx^m$  and the pseudo-energy is

$$E = \int dv \left[ \frac{1}{2 \cdot 4!} (T_{mnpq}^e T_{mnpq}^e + S_{mnpq} S_{mnpq}) + \frac{1}{2} (D_A \phi)_m^a (D_A \phi)_m^a + \lambda V(\phi) \right].$$

The Bogomol'nyi completion from two terms  $T^2$  and  $D\phi^2$  yields a topological term. This pseudo-energy is bound by the topological quantity  $E \geq \int_{S^4} \text{Tr} \phi F \wedge F = Q$ . Here  $S^4$  is the sphere of radius  $R$  which is large enough. We expect that the configuration localizes around a point and we take a limit  $R \rightarrow +\infty$ . For the field configuration which satisfy the Bogomol'nyi equation  $F \wedge F = \pm * D_A \phi$ , the pseudo-energy of this configuration attain the minimal

quantity  $Q$  after taking the Prasad-Sommerfeld limit  $\lambda \rightarrow 0$ . Let  $r^2 = x^a x_a$  be the radius and  $e = x^a \gamma_a / r$  be a unit vector. The solution of the Bogomol'nyi equation was studied with a hedgehog ansatz,  $\phi = H_0 U(r) e$  and  $A = (1 - K(r)) e d e / 2g$ , numerically. Here  $H_0$  is the absolute value of vacuum configuration of the scalar field  $\phi$ . Let  $\hat{x}_m = x_m / r$  be a unit vector of the radial direction. The differential form  $e d e$  is written in terms of these components;  $e d e = C_{mn} \gamma_{mn}$  where  $C_{mn} = (1/2)(\hat{x}_m d\hat{x}_n - \hat{x}_n d\hat{x}_m)$ . The expected boundary conditions are  $U(0) = 0, K(0) = 1, U(\infty) = \pm 1, K(\infty) = 0$ . The covariant derivative  $D_A \phi$  and the field strength are written in terms of  $K$  and  $U$ :  $D_A \phi = H_0 (U' e dr + K U d e)$ ,  $F = (1 - K^2) d e \wedge d e / 4g - K' e dr \wedge d e / 2g$ . Similary  $F^{\wedge 2}$  is rewritten as  $F \wedge F = ((1 - K^2)/4g)^2 d e^{\wedge 4} - ((1 - K^2)K'/4g^2) e dr \wedge d e^{\wedge 3}$ . The Hodge dual of these forms can be read from  $*(d e^{\wedge 4}) = 4! e dr / r^4$ ,  $*(e dr \wedge d e^{\wedge 3}) = 3! d e / r^2$ . The Bogomol'nyi equations reduce to a system of ordinary differential equations;  $(4!/r^4)((1 - K^2)/4g)^2 = H_0 U'$ ,  $-(3!/r^2)(1 - K^2)K'/4g^2 = H_0 K U$ . We scale the radial coordinate  $r$  by a factor  $a = (2g^2 H_0 / 3)^{1/3}$ ,  $\tau = ar$ .

$$\frac{dU}{d\tau} = \frac{1}{\tau^4} (1 - K^2)^2, \quad -\frac{1}{\tau^2} (1 - K^2) \frac{dK}{d\tau} = K U. \quad (1)$$

Let us study the behavior of functions  $K, U$  around boundaries. Let us consider the boundary  $\tau = 0$ . Suppose that  $K(\tau) = 1 + V(\tau)$ . Then  $V(\tau)$  and  $U(\tau)$  are small enough around  $\tau \sim 0$ . We drop terms  $V^3, VU$  and so on. From the equation (1), We obtain  $(dU/d\tau) = 4V^2/\tau^4$ ,  $(dV^2/d\tau)/\tau^2 = U$ . It implies that  $U \sim \tau$  and  $V \sim \tau^2$ . Let us consider  $\tau = \infty$ . The differential  $dU/d\tau$  should be positive and the initial value  $U(0)$  is zero. It means that the preferable boundary condition is  $U(\infty) = 1$ . Let us put  $U = 1 + W$ .  $W' \sim 1/\tau^4$ ,  $-(1/\tau^2)K' \sim K$ . Their solutions are  $W \sim 1/\tau^3$  and  $K \sim \exp(-\tau^3/3)$ . This system can be reduced to the first order ordinary equation. Let us change variables  $s = \ln \tau$  ( $-\infty < s < +\infty$ ),  $x(s) = K^2$  and let us eliminate the function  $U$ . Boundary conditions become  $x(-\infty) = 1, x(+\infty) = 0$ ,  $\dot{x}(-\infty) = 0, \dot{x}(+\infty) = 0$ , where we denote the derivative,  $dx/ds$ , " $\dot{x}$ ". After that we obtain an autonomous equation;

$$\frac{d^2 x}{ds^2} - \frac{1}{x(1-x)} \left( \frac{dx}{ds} \right)^2 - 3 \frac{dx}{ds} + 2x(1-x) = 0. \quad (2)$$

This equation does not include the variable  $s$  explicitly, (i.e. it is autonomous) and we can reduce this equation to the first order differential equation of  $y(x) = (dx/ds)/x(1-x)$ . The boundary conditions are  $(x, y) = (1, -2)$  at  $\tau \sim -\infty$  and  $(x, y) = (0, \infty)$  at  $\tau \sim +\infty$ .

$$x(1-x)y \frac{dy}{dx} = 2xy^2 + 3y - 2. \quad (3)$$

This equation (3) is an Abel differential equation of the second kind [10, 11] and is an algebraic differential equation. This equation does not survive from the Painlevé's theorem and the equation might have a movable branch point (Fig.1).

Here we use the fourth order Runge-Kutta method and start from  $x = 1/2$ . The initial value is  $y(1/2)$  in this case. The boundary  $(1, -2)$  exists in finite range, while the boundary  $(0, \infty)$  does not. Let us show flows around the boundary  $(1, -2)$  (Fig.2). Fig.3 shows that the initial value which gives a flow converging to the point  $(1, -2)$  is  $-2.9222$ . In order to clarify the behavior around  $(0, \infty)$  let us consider a transformation. Let us rewrite  $y(x)$  in terms of  $K, U, \tau$ . From (1), we obtain

$$y(x) = \frac{\tau dx/d\tau}{x(1-x)} = -\frac{2\tau^3 U}{(1-K^2)^2}.$$

This implies that  $y(x)$  behave like  $\ln x$  around the boundary  $\tau \sim +\infty$  because  $K(\tau) \sim \exp(-\tau^3/3)$  and  $U(\tau) \sim 1$  ( $1 < \tau$ ). Therefore let us put  $y(x) = (\ln x)u(x)$ .

$$x(1-x)(\ln x)u \frac{du}{dx} = \{2x(\ln x) - (1-x)\}u^2 + 3u - \frac{2}{\ln x}. \quad (4)$$

Suppose that in the limit  $x \rightarrow 0$ ,  $u(x)$  and  $(du/dx)(x)$  have finite limits;  $-u^2 + 3u \rightarrow 0$ ,  $(\tau \rightarrow +\infty)$ ,  $u(+\infty) = 0, 3$ . In fact, Fig. 4 shows that  $u(+\infty) = 3$ . This evaluation showed us the existence of the solution.

Let us rewrite the differential equation (3) in terms of the homogeneous coordinate  $[X : Y : Z : W]$ ;  $x = X/W$ ,  $y = Y/W$ ,  $(dy/dx) = Z/W$ . Then they satisfy an algebraic relation  $F(X, Y, Z, W) := X(W-X)YZ - 2XY^2W - 3YW^3 + 2W^4 = 0$ . This relation defines a projective variety in  $\mathbb{CP}^3$  and the degree is 4. The variety is singular and it is not a K3 surface. We wonder if there exist another solutions which relate to K3 surfaces.

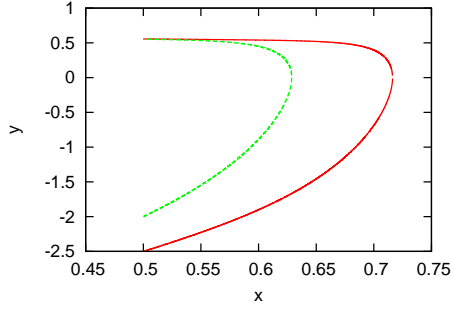


FIG. 1: Movable branch point.

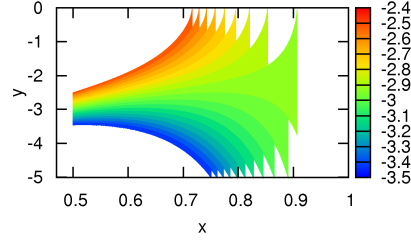


FIG. 2: Flows near  $(1, -2)$  in the  $xy$ -plane.

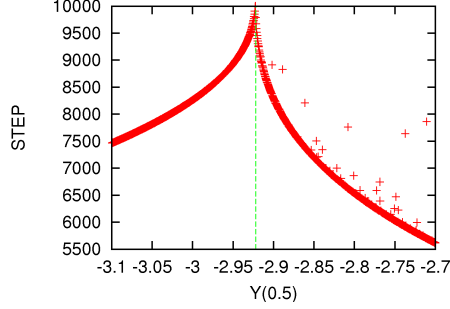


FIG. 3: Numbers of steps in the Runge-Kutta method with thresholds ( $\text{const.} < y < 0$ ).

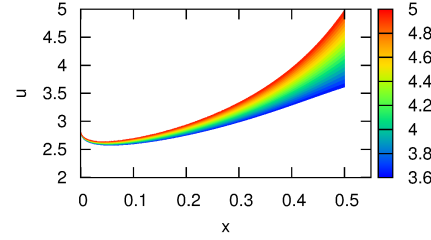


FIG. 4: Flows near  $(0, 3)$  in the  $xu$ -plane.

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